International Journal of Theoretical Physics, Vol. 23, No. 8, 1984

Geometric Quantization in Many-Body Physics

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Received October 1, 1983

The Bohr–Mottelson model of nuclear rotations and vibrations is a cornerstone of nuclear structure physics (Bohr et al., 1976). The model regards the nucleus as a liquid drop deformed into an ellipsoid which rotates and incompressibly vibrates, thereby giving rise to rotational bands and strong electric quadrupole transitions. By experiment, one can determine the shape from the intrinsic quadrupole moments and measure the nuclear moment of inertia from the energy levels. The success of this model stems not only from its generally favorable agreement with experiment, but also from the transparent geometrical view it provides of the nucleus.

Nevertheless, the Bohr–Mottelson model has inherent limitations. In order to explain detailed nuclear properties it is obvious that we need to incorporate noncollective features of the nucleus, e.g., shell structure, into our model. But, how can we relate the liquid drop picture to the microscopic view of the nucleus as an interacting system of neutrons and protons? At the 1970 Solvay Conference, Professor Wigner emphasized the importance of answering this question for nuclear structure physics (Wigner, 1970). We need to be able to pose and solve nuclear structure problems with as much precision and detail as is necessary, but not at the expense of obscuring the fundamental geometrical properties of the nucleus.

The resolution of this dilemma involves ideas from geometric quantization and dynamical groups which have parallels with the relativistic free particle. As we shall see, the absence of a theory of interacting relativistic particles does not allow us to pursue the correspondence to its ultimate conclusion. However, we will succeed in embedding the liquid drop model into the existing theory of interacting nonrelativistic neutrons and protons, thereby resolving the Solvay question.

0020-7748/84/0800-0777$03.50/0

GEOMETRIC QUANTIZATION

Geometric quantization has achieved significant progress toward a clear understanding of the relationship between classical and quantum mechanics (Kostant, 1970; Souriau, 1970). The theory formulates a definition of quantization suitable for arbitrary symplectic manifolds by identifying the intrinsic geometric objects involved in ordinary Dirac quantization. Furthermore, there is a classification theorem for the symplectic manifolds on which a Lie group acts transitively and canonically in terms of the group's co-adjoint orbits.

These ideas coalesce in a striking application to the relativistic free particle. One defines a classical model for a free particle as a symplectic manifold on which there is a transitive action of the Poincaré group by canonical transformations. Since these phase spaces are naturally given as co-adjoint orbits of the Poincaré group, all the possible classical models for a free relativistic particle are determined. It is found that there is a phase space for every possible mass and also for every possible spin! This is in contrast to the conventional models given by the cotangent bundles of orbits of the Lorentz group in Minkowski energy-momentum space (mass hyperboloids), which describe only spinless particles.

Curiously, this general classification principle was first employed in its quantum setting. In a landmark study, Wigner (1939) characterized the quantum state space of a free particle as a Hilbert space which carries an irreducible unitary representation of the Poincaré group. Using the inducing construction, all possible quantum mechanical state spaces are enumerated. Representations with arbitrary mass and half-integral spin are obtained. Of course, ordinary Dirac quantization of the mass hyperboloids omits the nonzero spin possibilities.

The general relationship between the classical symplectic spaces and the quantum Hilbert spaces is that the quantum irreducible representations are given by quantization of the classical co-adjoint orbits. This quantization is the Kostant–Souriau construction suitable for arbitrary symplectic manifolds meeting the generalized Bohr–Sommerfeld quantization conditions.

With this overview of the relativistic free particle in mind, let us survey the parallels with the liquid drop model.

CLASSICAL COLLECTIVE MODELS

In the liquid drop model, the classical configuration space is an orbit of the special linear group $SL(3)$ in the space Q of three-by-three, real symmetric positive-definite matrices (Rosensteel and Rowe, 1979). This parallels the orbits of the Lorentz group in Minkowski energy-momentum space. A point q of Q is physically identified with the mass quadrupole–monopole moment of a classical fluid, $q_{ij} = \int \rho(x) x_i x_j d^3x$, where ρ is the density distribution of the fluid. The action of $SL(3)$ on Q is inherited from its natural action on three-dimensional Euclidean space, $q \to g \cdot q \cdot {}^t g$, for $g \in SL(3)$. The constraint to an orbit of the volume-preserving transformations insures the incompressibility of the fluid. Each orbit in Q space corresponds to a different nuclear volume. The cotangent bundles to these orbits are the phase spaces of a liquid drop.

However, in order to characterize all possible collective models we need a Lie group for the liquid drop which parallels the Poincaré group. The group is known as $CM(3)$, which stands for collective motion in three dimensions. It is a semidirect product of $SL(3)$ with the six-dimensional abelian normal subgroup generated by the translations in Q space.

A classical collective model is defined now to be a symplectic manifold with a transitive action of $CM(3)$ by canonical transformations (Rosensteel and Ihrig, 1979; Guillemin and Sternberg, 1980). These models are exhausted by the co-adjoint orbits of $CM(3)$. The orbits are indexed by two real parameters (λ, v), where $\lambda^3 = \det q$ measures the volume and v determines the total vortex momentum. The singular orbits with vanishing vortex momentum (irrotational flow) are just the cotangent bundles of orbits of $SL(3)$ in Q space. However, to the author's knowledge the nonzero vortex generic orbits are new classical models for incompressible fluids.

QUANTUM COLLECTIVE MODELS

Just as Wigner characterized the quantum state space for a free particle in terms of the irreducible representations of the Poincaré group, we define a quantum collective model as an irreducible representation of $CM(3)$ (Rosensteel and Rowe, 1976; Weaver, Cusson and Biedenharn, 1976). Since $CM(3)$ is a semidirect product with an abelian normal subgroup just like the Poincaré group, its irreducible unitary representations are all given via the inducing construction.

The irreducible unitary representations of $CM(3)$ are indexed by two parameters (λ, L), where again $\lambda^3 = \det q$ measures the volume and L is an integer with $\mathsf{L}(\mathsf{L}+1)$ the total vortex momentum. The representations with vanishing vortex momentum are the spaces of the Bohr–Mottelson irrotational flow model. However, the nonzero vortex spin representations represent new quantum fluid models.

What is the relationship between the classical and quantum collective models? For the vortex free models, the quantum space is given by ordinary Dirac quantization of the classical configuration space, viz. an orbit of $SL(3)$ in Q space. This quantization was carried out by A. Bohr (1952). However, for the models with nonvanishing vortex momentum, we must be careful. Only the co-adjoint orbits of $CM(3)$ which satisfy the (Niels) Bohr–Sommerfeld quantization condition can be quantized. The allowed co-adjoint orbits have $2\pi v$ integral. This should be compared to the restriction to half-integral spin orbits for the Poincaré group, which Souriau found necessary for the quantization of the free particle. After applying the Kostant–Souriau quantization construction to the co-adjoint orbit (λ, v) with $2\pi v$ integral, we obtain the irreducible unitary representation (λ, L) of $CM(3)$, where $\mathsf{L} = 2\pi v$. Hence, every quantum collective model arises from the quantization of some classical model.

INTERACTING SYSTEMS

So far, the correspondence between the relativistic free particle and the quantum liquid drop has been remarkable. For each theoretical concept relevant to the relativistic free particle, an analogous idea applies to the liquid drop. However, we can carry out the theory of a liquid drop one step further, which, although desirable, has no analog for the free particle. The quantum collective models can be naturally embedded in the complete interacting theory of A protons and neutrons. But, since we do not know what the theory of interacting relativistic particles is, no such embedding is possible for free particles.

The Hilbert space of the quantum nuclear A particle system is the exterior product of A copies of the single-particle space. The group $CM(3)$ acts reducibly on this Hilbert space. In the reduction into irreducible representations, it is found that every irreducible representation of $CM(3)$ occurs with countably infinite multiplicity. Hence, every quantum model of a liquid drop, including the nonvanishing vortex momentum representations, can occur in nature. An irreducible representation is seen experimentally if the observed quantum states are selected from a single irreducible representation. This favorable selection will only occur if the irreducible subspace is invariant with respect to the exact nuclear Hamiltonian.

We would like to be able to carry out a similar analysis for the Poincaré group, but, of course, we need to know what the action of the Poincaré group is on the interacting relativistic particle space. We would then reduce the representation of the Poincaré group on the interacting Hilbert space into its irreducible components. Professor Dirac in his talk conjectured that the action of the Poincaré group on the interacting space is not reducible and indeed we should be looking for such pathological representations (in Mackey's classification scheme, these are type-III representations). It remains to be seen what can be done here.

The classical collective models are realized on A particle phase space R^{6A} by employing the moment map, which naturally maps R^{6A} into the dual space of the Lie algebra of $CM(3)$. The group $CM(3)$ acts as canonical transformations on R^{6A}. The moment map intertwines the group action on R^{6A} and the co-adjoint action on the dual space. Hence, an orbit of $CM(3)$ in R^{6A} is mapped by the moment map onto a co-adjoint orbit.

Apparently, this achieves a decomposition of R^{6A} into collective submanifolds. However, there is a difficulty. In general, the orbits of $CM(3)$ in R^{6A} are not symplectic manifolds. This is reflected in the fact that the restriction of the moment map to an orbit of $CM(3)$ in R^{6A} in not a one-to-one map onto its corresponding co-adjoint orbit. Since the moment map is not injective, it does not necessarily follow that the integral curves of a Hamiltonian vector field, even though lying entirely in an orbit, will factor through the moment map. Further work is needed on quotienting out integrals via the moment map.

ACKNOWLEDGMENTS

I would like to acknowledge the work of my collaborators, Ed Ihrig and David Rowe, and thank Bert Kostant and Shlomo Sternberg for stimulating this application of geometric quantization to many-body physics.

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